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The expected uncertainty of range-free localization protocols in sensor networks

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Abstract

We consider three range-free localization protocols for sensor networks and analyze their accuracy in terms of the expected area of uncertainty of position per sensor. Assuming a small set of anchor nodes that know their position and broadcast it, we consider at first the simple Intersection protocol. In this protocol a sensor assumes its position is within the part of the plane that is covered by all the broadcasts it can receive. We then extend this protocol by assuming every sensor is preloaded with the entire arrangement of anchors before being deployed. We show that in this case the same expected uncertainty can be achieved with $1/2$ the anchor nodes. Finally, we propose an approximation for the arrangement-based protocol which does not require any preliminary steps and prove that its expected accuracy converges to that of the arrangement protocol as the number of anchors increases.

Keywords: Range free localization; Sensor networks

1. Introduction

Sensor networks provide wireless connectivity for stationary sensors that are usually embedded in some random fashion within a physical domain. The sensors' purpose is to monitor and report locally sensed events so it is common to assume that a sensor can position itself within some global coordinate system. Since the standard Global Positioning System is considered too costly for deployment within every sensor, other localization mechanisms

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specific for sensor networks are typically used. Such mechanisms can be divided into *range-based* systems and *range-free* systems.

In range-based systems every sensor is augmented with hardware that is specifically necessary for localization. For example, augmented sensors may be able to deduce their distance from each other. This information is extremely helpful for the localization process and such systems can generally achieve fine grained localization. However, augmenting the hardware of every sensor in this way may be very costly. Range-free systems, on the other hand, attempt to reduce the number of augmented sensors to a minimum. A small subset of the sensors is augmented, typically with full GPS positioning systems. These sensors (called *anchors*) broadcast their position and the other sensors use this information to localize themselves. The most important success measure for such systems is the localization error, given a small, fixed number of anchors.

Although many range-free localization systems were suggested in the past [7,6,2,9,4], not a lot has been done in terms of analytically evaluating their expected error. In this paper we consider three range-free localization protocols starting with the basic intersection protocol described in [10], and analyze their expected accuracy in terms of the area of uncertainty of position per sensor. As we show, simple enhancements to the basic protocol produce significant reductions in the expected area of uncertainty.

Assume that N sensors are randomly placed in the unit disk and that only a subset of size $K < N$ of these sensors (the *anchors*) know their exact position in terms of some global coordinate system. Have these anchors broadcast their position and let that broadcast be received by and only by sensors that are within the *communication range*, ρ, which defines a disk of radius ρ centered at the broadcasting anchor; see Fig. 1(a). Denoting as *neighbors* sensors that can directly communicate with each other we describe three localization protocols. In the *Intersection* protocol, a sensor assumes its position is within the intersection of the communication ranges of its neighbor anchors; see Fig. 1(b). In the *Arrangement* protocol, a sensor assumes its position is at the intersection of its neighbor anchors that is outside the range of any other anchor; see Fig. 1(c). Finally, in the *Approximate Arrangement* protocol, a sensor assumes its position is anywhere within range of its neighbor anchors that is out of range of its neighbor-neighbors anchors, see Fig. 1(d).

We define the *uncertainty* of position for a sensor to be the area size of the possible locations for it and present analytic expressions for the expected uncertainty in the Intersection and Arrangement protocols. Furthermore, we show that the Arrangement protocol reaches the same expected uncertainty as the Intersection protocol with at most $1/2$ of the anchors. We then consider the approximation protocol, showing both analytically and by simulations that its expectation converges to that of the Arrangement protocol.

1.1. Related work

Numerous range-free localization mechanisms have been proposed. The first solutions were typically central. For example, the connectivity matrix of the sensors is used in [3] to constrain the possible locations of the sensors in the network, achieving a position estimation by centrally solving a convex optimization problem. Many distributed solutions were also suggested. In the Centroid strategy, sensors locate themselves at the center of mass

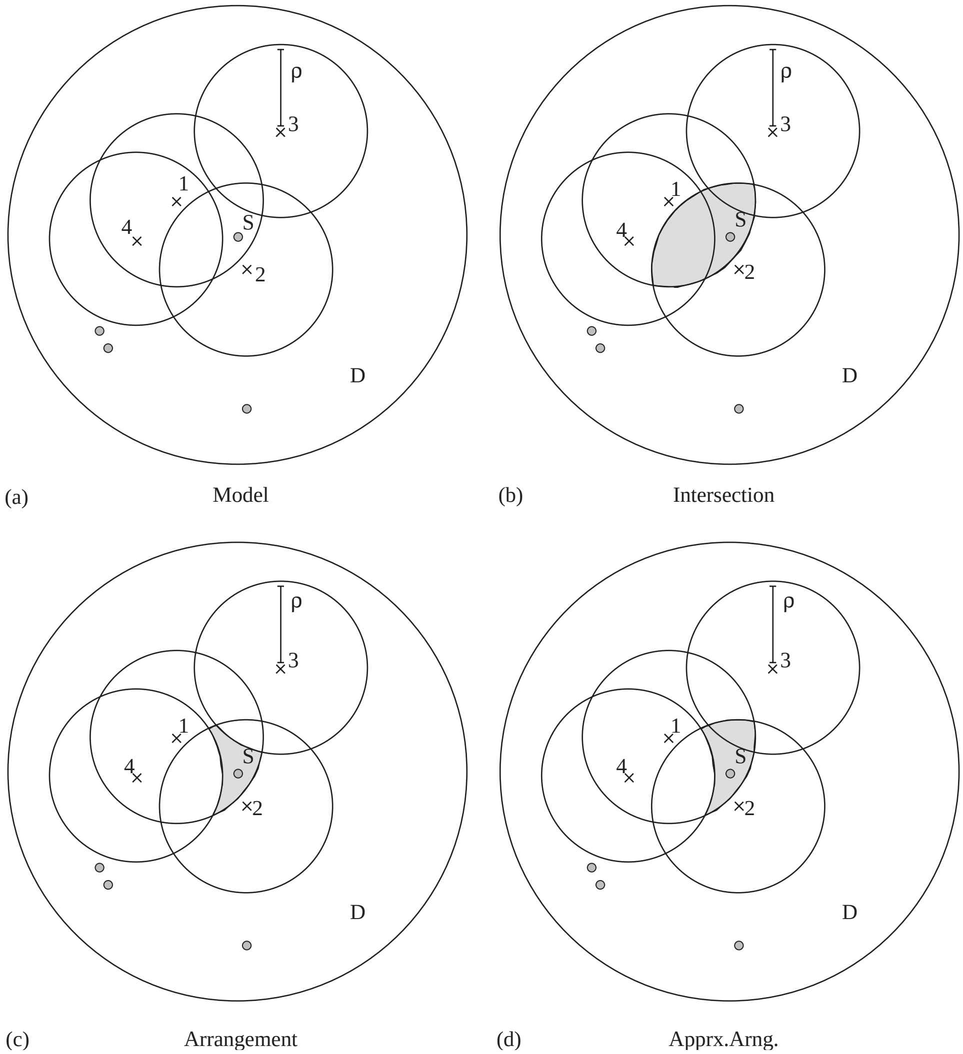

Fig. 1. Four anchors (numbered crosses) and several other sensors (bullets) are scattered in the unit disk $\mathbb{D}$. Anchor x_i broadcasts its coordinates in the disk, $D(x_i, \rho)$, that is centered at x_i and has radius ρ (a). Sensor s can use the Intersection protocol (b) to position itself at the intersection of the disks related to anchors within range (1 and 2). It can use the Arrangement protocol (c) to position itself at the intersection of these disks that is not within range of any of the other anchors, 3 and 4. Or it can use the approximation (d) to locate itself in a slightly bigger region, defined in this case by the intersection of the ranges of anchors 1 and 2 that is not in the range of anchor 4 (anchor 3 is ignored because it is not a neighbor of the neighbors of s).

(the centroid) of the locations of anchors they can detect [2,1]. Other approaches [7,6] assumed node to node communication can be used to flood the location of all the anchors to all the sensors. Using the hop-count as an estimate of the Euclidian distance (by computing the average distance between sensors [7] or by analytically deriving it [6]), a sensor can estimate its position via triangulation. The accuracy of all of these solutions, however, was evaluated experimentally, rather than analytically.

The research of range-based systems has been more extensive, supplying bounds on the achievable positioning accuracy when assuming specific calibration data, noise patterns and location distribution. In particular, lower bounds for the localization uncertainty using the Cramér–Rao bound were presented in [11,8]. Still, the behavior of the specific algorithms is almost always investigated by simulation, either by the authors or in subsequent works [5].

1.2. Paper organization

Section 2 describes the model we use. In Section 3 we analyze the expected uncertainty of the Intersection protocol. In Section 4 we analyze the expected uncertainty of the Arrangement protocol, comparing it with the results of Section 3. In Section 5 we describe the approximation protocol and show that its expected accuracy converges to that of the Arrangement protocol. We conclude the paper with Section 6.

2. Model

For any point p in the unit disk $\mathbb{D}$ and for any constant, ρ, denote by $D(p, \rho)$ the disk centered at p and of radius ρ (if p is the origin we simply write $D(\rho)$). To emphasize that a particular disk contains only the points in $\mathbb{D}$ that are at least ρ away from the boundary $\partial\mathbb{D}$ we write $\mathbb{D}_\rho := D(1 - \rho)$.

Generally we assume that N sensors, $S_1, \ldots, S_N$, are placed by choosing the location of each independently from the uniform distribution over the unit disk $\mathbb{D}$ in the plane. Any two sensors p_i, p_j, can communicate with each other only if the Euclidean distance between them is less than or equal to the *communication range*, $\rho < 1/3$, $\|p_i - p_j\| \leqslant \rho$. In this case we say that the two sensors are *within range* of each other and consider them to be *neighbors*.

Of the N sensors, only K know their coordinates. We name such sensors *anchors* and denote the set of anchors by $\mathcal{S}_A$. We will often abuse notation and consider $\mathcal{S}_A$ to be the set of the point locations of the sensors.

To avoid boundary effects we assume that while the anchors are randomly placed in all of $\mathbb{D}$, the other $N - K$ sensors are randomly placed in the smaller region $\mathbb{D}_\rho$. It is easy to see that without this assumption sensors close to the boundary of $\mathbb{D}$ would have lower probability of having anchor neighbors. Moreover, for simplicity it is necessary to restrict the expectation analysis to the sensors that are in $\mathbb{D}_{3\rho}$. For consider the case where a sensor S has exactly one neighbor anchor that is positioned less than ρ away from the boundary of $\mathbb{D}_\rho$. Obviously, in this case the set of possible locations for S depends on the exact distance of its neighbor from the boundary.

3. Localization by intersection

The simplest localization approach for a sensor is to collect the positions of all its neighbor anchors and place itself somewhere at the intersection of the communication ranges of all these anchors, see Fig. 1(b). This is probably the most direct localization mechanism and it has been studied in [10]. Their analysis, however, is wrong since it neglected to take into account all the possible scenarios (for example, the case $\mathcal{S}_N = \emptyset$). In this section we introduce a new and simpler approach for the analysis of the expected size of this intersection, achieving correct results as opposed to [10]. Furthermore, we use the standard $\mathbb{R}^2$ space with L_2 norm, rather than the discrete L_∞ model used in [10].

Let S be a randomly picked sensor and denote by $\mathcal{S}_N$ the set of its anchor neighbors.

$$\mathcal{S}_N = \{s \in \mathcal{S}_A : \|s - S\| \leqslant \rho\}. \tag{1}$$

Denote by I_S the area of possible locations for S

$$I_S = \begin{cases} \bigcap_{s \in \mathcal{S}_N} D(s, \rho) & \text{if } \mathcal{S}_N \neq \emptyset, \\ \mathbb{D}_\rho & \text{otherwise.} \end{cases}$$

That is, I_S is the area of the intersection of the communication ranges of the neighbors of S if there are any, or the entire set of possible places, $\mathbb{D}_\rho$, otherwise.

Let $X = |I_S|$ be the area size of I_S. Then X is a random variable, $0 \leqslant X \leqslant \pi(1-\rho)^2$ and its expectation, $\mathrm{E}(X)$, can be analyzed:

$$\mathrm{E}(X) = \int_{\mathbb{D}^K} \pi^{-K} \left(\int_{\mathbb{D}_\rho} 1_{I_S}(u)\, \mathrm{d}u \right) \mathrm{d}p_1\, \mathrm{d}p_2 \cdots \mathrm{d}p_K,$$

where $1_{I_S}(u)$ is the indicator function,

$$1_{I_S}(u) = \begin{cases} 1 & u \in I_S, \\ 0 & \text{otherwise.} \end{cases}$$

Rewriting the integration,

$$\begin{aligned} \mathrm{E}(X) &= \int_{\mathbb{D}_\rho} \left(\int_{\mathbb{D}^K} \pi^{-K} 1_{I_S}(u)\, \mathrm{d}p_1\, \mathrm{d}p_2 \cdots \mathrm{d}p_K \right) \mathrm{d}u \\ &= \int_{\mathbb{D}_\rho} \Pr[u \in I_S]\, \mathrm{d}u. \end{aligned}$$

For any sensor $S \in \mathbb{D}_{3\rho}$ denote by $\mathbf{E}_1$ the event where $u \notin D(S, 2\rho)$ and by $\mathbf{E}_2$ the event where $u \in D(S, 2\rho)$. Avoiding boundary affects by restricting the analysis to $\mathbb{D}_{3\rho}$ we have

$$\mathrm{E}(X|S \in \mathbb{D}_{3\rho}) = \int_{\mathbb{D}_\rho \setminus D(S,2\rho)} \Pr[u \in I_S|\mathbf{E}_1]\, \mathrm{d}u + \int_{D(S,2\rho)} \Pr[u \in I_S|\mathbf{E}_2]\, \mathrm{d}u.$$

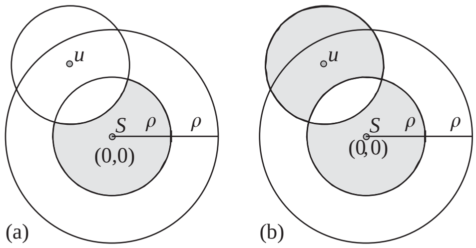

Fig. 2. For the Intersection protocol point u, $0 \leqslant \|u\| \leqslant 2\rho$ is in I_S if none of the K anchors fall in the shaded area in (a). For the Arrangement protocol point u, $0 \leqslant \|u\| \leqslant 2\rho$ is in A_S if none of the K anchors fall in the shaded area in (b).

If u is outside $D(S, 2\rho)$ then it is in I_S only if $\mathcal{S}_N = \emptyset$ (S has no neighbor anchors). Thus,

$$\Pr[u \in I_S|\mathbf{E}_1] = \Pr[u \in I_S|u \notin D(S, 2\rho) \wedge S \in \mathbb{D}_{3\rho}] = \left(1 - \rho^2\right)^K$$

and therefore

$$\mathrm{E}(X|S \in \mathbb{D}_{3\rho}) = \left(1 - \rho^2\right)^K \left(\pi(1 - \rho)^2 - 4\pi\rho^2\right) + \int_{D(S,2\rho)} \Pr[u \in I_S|\mathbf{E}_2]\,\mathrm{d}u.$$

On the other hand, point $u \in D(S, 2\rho)$ is in I_S only if every anchor in the range of S is also in the range of u. Hence, u is in I_S only if all the K anchors fall outside $D(S, \rho) \setminus D(u, \rho)$; see Fig. 2(a).

Notice that the two disks intersect in a lens-shaped region, l, whose size $|l|$ depends only on $\|u - S\|$. By moving the origin to S and changing to polar coordinates we have,

$$\begin{aligned}\mathrm{E}(X|S \in \mathbb{D}_{3\rho}) = &\left(1 - \rho^2\right)^K \left(\pi(1 - \rho)^2 - 4\pi\rho^2\right)\\ &+2\pi \int_{r=0}^{2\rho} \left(1 - \rho^2 + \pi^{-1}|l(r)|\right)^K r\,\mathrm{d}r.\end{aligned}$$

Let A, C be the two points located at the cusps of the lens $l(r)$, see Fig. 3. Denote by B the intersection point of the chord AC with the segment Su that connects the two disk origins and mark by α the angle $\angle ASB$. Then the area of $l(r)$ is $|l(r)| = 2\rho^2\alpha - \rho r \sin\alpha$, where $r = 2\rho\cos\alpha$, so

$$|l(r)| = 2\rho^2 \arccos(\tfrac{r}{2\rho}) - \rho r\sqrt{1 - (\tfrac{r}{2\rho})^2}. \tag{2}$$

Finally, we have

$$\begin{aligned}\mathrm{E}(X|S \in \mathbb{D}_{3\rho}) = &\left(1 - \rho^2\right)^K \left(\pi(1 - \rho)^2 - 4\pi\rho^2\right)\\ &+2\pi \int_{r=0}^{2\rho} \left(1-\rho^2+\tfrac{2}{\pi}\rho^2 \arccos(\tfrac{r}{2\rho})-\tfrac{\rho}{\pi}r\sqrt{1-(\tfrac{r}{2\rho})^2}\right)^K r\,\mathrm{d}r.\end{aligned} \tag{3}$$

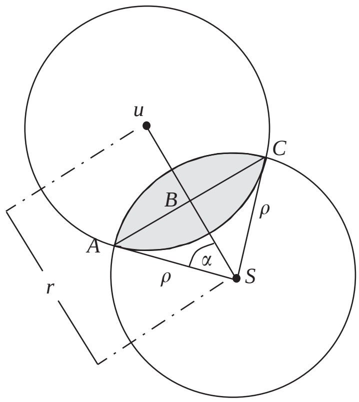

Fig. 3. The size of the lens (shaded area) created by the intersection of the two disks $D(S, \rho)$ and $D(u, \rho)$ is $|l(r)| = 2\rho^2\alpha - \rho r \sin\alpha$.

4. Localization by arrangement

It is possible to improve the basic Intersection protocol if we assume that every sensor knows the position of all the K anchors (rather than just its neighbors). This information can be gathered by flooding it over the network or by placing the sensors in several phases. With this knowledge a sensor can position itself at the intersection of the communication range of its anchor neighbors that is not within range of any other anchor; see Fig. 1(c). In this section we analyze the expected uncertainty of the Arrangement protocol and show that it achieves the same expected uncertainty as the Intersection protocol using at most half the anchors.

As before, let S be a randomly picked sensor, $\mathcal{S}_N$ be the set of its anchor neighbors and denote by $\mathcal{S}_N^c$ all the other anchors, $\mathcal{S}_N^c = \mathcal{S}_A \setminus \mathcal{S}_N$. Let A_S be the set of all the possible places for S,

$$A_S = \begin{cases} \bigcap_{s\in\mathcal{S}_N} D(s,\rho) \setminus \bigcup_{s\in\mathcal{S}_N^c} D(s,\rho) & \text{if } \mathcal{S}_N \neq \emptyset, \\ \mathbb{D}_\rho \setminus \bigcup_{s\in\mathcal{S}_N^c} D(s,\rho) & \text{otherwise.} \end{cases} \tag{4}$$

That is, if S has neighbors then A_S consists of all the points in $\mathbb{D}_\rho$ that are in range of the neighbor anchors of S and out of range of all the other anchors. If S has no neighbor anchors then A_S consists of all the points in $\mathbb{D}_\rho$ that are outside the range of all the anchors. [1]

Following the proof strategy and notation of Section 3, we define $\mathbf{E}_1$ to be the event that $u \notin D(S, 2\rho)$, $\mathbf{E}_2$ to be the event that $u \in D(S, 2\rho)$ and consider only cases where

[1] It should be noted that A_S does not have to be convex, or even connected (consider e.g. configurations where $\mathcal{S}_N = \emptyset$).

$S \in \mathbb{D}_{3\rho}$. Denoting $X = |A_S|$ the area of A_S we have

$$\mathrm{E}(X|S \in \mathbb{D}_{3\rho}) = \int_{\mathbb{D}_\rho \setminus D(S,2\rho)} \Pr[u \in A_S|\mathbf{E}_1]\,\mathrm{d}u + \int_{D(S,2\rho)} \Pr[u \in A_S|\mathbf{E}_2]\,\mathrm{d}u.$$

Now point $u \notin D(S, 2\rho)$ can be in A_S only if both S and u have no anchor neighbors,

$$\Pr[u \in A_S|\mathbf{E}_1] = \Pr[u \in A_S|u \notin D(S,2\rho) \wedge S \in \mathbb{D}_{3\rho}] = \left(1 - 2\rho^2\right)^K.$$

Point $u \in D(S, 2\rho)$ is in A_S only if all the K anchors fall outside $(D(S, \rho) \cup D(u, \rho)) \setminus (D(S, \rho) \cap D(u, \rho)$; see Fig. 2(b). Thus, denoting by r the distance of u from S, $r = \|u - S\|$, we have

$$\begin{aligned}\Pr[u \in A_S|\mathbf{E}_2] &= \Pr[u \in A_S|u \in D(S,2\rho) \wedge S \in \mathbb{D}_{3\rho}] \\ &= \left(1 - 2\rho^2 + 2\pi^{-1}|l(r)|\right)^K,\end{aligned}$$

where $|l(r)|$ is the area of the intersection of the two disks given in (2). Assuming S is at the origin and changing to polar coordinates we finally have

$$\begin{aligned}\mathrm{E}(X|S \in \mathbb{D}_{3\rho}) = &\left(1 - 2\rho^2\right)^K \left(\pi(1-\rho)^2 - 4\pi\rho^2\right) \\ &+2\pi \int_{r=0}^{2\rho} \Big(1 - 2\rho^2 + 2\tfrac{2}{\pi}\rho^2 \arccos(\tfrac{r}{2\rho}) \\ &-2\tfrac{\rho}{\pi} r \sqrt{1 - (\tfrac{r}{2\rho})^2}\Big)^K r\,\mathrm{d}r, \qquad (5)\end{aligned}$$

where the emphasized factors of 2 are the only difference in the expression from the expectation of the Intersection protocol.

4.1. Arrangement vs. intersection

It is not too difficult [2] to see that asymptotically both (3) and (5) behave like $\mathrm{O}(1/K^2)$. The asymptotic behavior, however, is not particularly helpful since we are interested in the cases where K is small.

Denote by E_I^K (E_A^K) the expectation of the area of uncertainty of the Intersection protocol (Arrangement protocol) using K anchors and a constant ρ.

Let $c > 1$ be some constant s.t. $E_A^K \leqslant E_I^{(cK)}$. Denoting

$$x_r = \rho^2 - \pi^{-1}|l(r)|, \quad 0 \leqslant r \leqslant 2\rho,$$

we require c to be such that for all r, $(1-2x_r)^K \leqslant (1-x_r)^{cK}$. Noticing that $0 \leqslant x_r \leqslant \rho^2 < 0.5$, it is not difficult to see that

$$(1 - 2x)^K \leqslant (1 - x)^{2K} \quad 0 < x < 0.5,$$

so that $c = 2$ will do. As can be seen in Fig. 4, the constant is almost precisely two for any reasonable choice of ρ.

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[2] The first term of (3), (5) diminishes exponentially with K for any constant ρ. The second term is bounded by $\int_{r=0}^{1} c_1 r(c_2 - c_3 r)^K \,\mathrm{d}r = \mathrm{O}(1/K^2)$, for the appropriate constants.

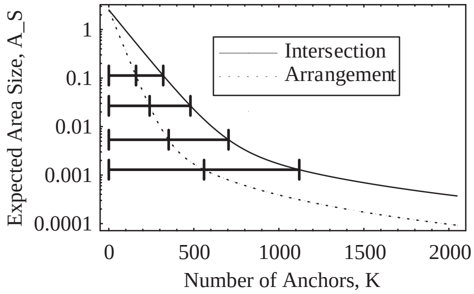

Fig. 4. A linear-log graph of the expected area size for the Intersection and Arrangement protocols ($\rho = .1$). The horizontal bars are aligned with the graph corresponding to the Intersection protocol and mark the middle point in terms of K. As can be seen, the Arrangement protocol achieves the same expected accuracy with almost precisely $\frac{1}{2}K$ anchors.

5. Approximating the arrangement

To consider a practical approximation for the arrangement protocol we assume that anchors can listen to the broadcasts of other anchors. Although at first anchors can only broadcast their own position, after an initial step every anchor can learn the positions of its neighbor anchors and broadcast the information along its own position. A sensor can thus create two disjoint lists: a list of its neighbors and a list of its neighbor-neighbors that are not also its own neighbors. A sensor can now position itself at the intersection of its neighbors that is not within range of any of its neighbor-neighbors, see Fig. 1(d). As we show, the accuracy of this protocol converges to the accuracy of the arrangement protocol when the number of anchors increases.

Let S be a randomly picked sensor. Denote as usual by $\mathcal{S}_N$ the set of its anchor neighbors and by $\mathcal{S}_{NN}$ the set of anchors which are not neighbors of S by themselves, but which have neighbors out of $\mathcal{S}_N$.

$$\mathcal{S}_{NN} = \{s \in \mathcal{S}_N^c : \exists s' \in \mathcal{S}_N \text{ s.t. } \|s - s'\| \leqslant \rho\}.$$

Denote by A_S^{aprx} the set of possible places for S. Then

$$A_S^{\text{aprx}} = \begin{cases} \bigcap\limits_{s \in \mathcal{S}_N} D(s, \rho) \setminus \bigcup\limits_{s \in \mathcal{S}_{NN}} D(s, \rho) & \text{if } \mathcal{S}_N \neq \emptyset, \\ \mathbb{D}_\rho & \text{otherwise.} \end{cases} \tag{6}$$

That is, if S has neighbors then A_S^{aprx} consists of the places that are within the range of the neighbors of S and outside the range of its neighbor-neighbors. If S has no neighbors then A_S^{aprx} consists of all of $\mathbb{D}_\rho$.

When evaluating $\mathrm{E}(|A_S^{\text{aprx}}|)$, one can use the same type of arguments as in Section 3 for positions that are outside $D(S, 2\rho)$ (such positions are in A_S^{aprx} only if $\mathcal{S}_N = \emptyset$). However, for any point $u \in D(S, 2\rho)$, the expectation is conditioned on the number and position of anchors in $D(u, \rho) \cap D(S, \rho)$; see Fig. 5. It is not trivial to analytically express the expectation for this case. Instead, we show that for a large enough K the area of

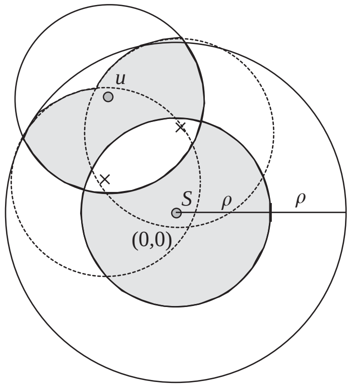

Fig. 5. Conditioned on the position of the two anchors in $D(S, \rho) \cap D(u, \rho)$ (marked by crosses), point u is in A_S^{aprx} if the other $K - 2$ anchors fall outside the shaded area.

uncertainty returned by the Approximation protocol is equal with high probability to the area of uncertainty returned by the original Arrangement protocol.

5.1. Convergence in probability

For any sensor S and any specific configuration of locations of anchors, denote by A_S the set returned by the Arrangement protocol as defined in (4) and by A_S^{aprx} the set returned by the Approximation protocol as defined in (6). Taking the definition of $\mathcal{S}_N$ and I_S from Section 3 and the definition of $\mathcal{S}_{NN}$ from Section 5 we define $\mathcal{S}_O$ to be the set of anchors that are not in $\mathcal{S}_N$ and yet are within range of some point in I_S,

$$\mathcal{S}_O = \{s \in \mathcal{S}_N^c : \exists u \in I_S \text{ s.t. } \|u - s\| \leqslant \rho\}.$$

Thus, $\mathcal{S}_O$ consists of all the anchors that help reduce the size of A_S from the size of I_S. As usual, we abuse notation and treat $\mathcal{S}_O$ also as a set of point locations.

Denote by B_W, B_E, B_N, B_S the four disjoint disks fully contained in $D(S, \rho)$ and of radius $c\rho$ for the constant $c = \frac{\sqrt{2}-1}{3\sqrt{2}} < 1/4$ which are placed in $D(S, \rho)$ at the leftmost, rightmost, topmost and bottommost possible positions, respectively; see Fig. 6(a). Thus, assuming as usual that S is at the origin we have $B_W = D(((c-1)\rho, 0), c\rho)$, $B_E = D((1-c)\rho, 0), c\rho)$, $B_N = D((0, (1-c)\rho), c\rho)$, $B_S = ((0, (c-1)\rho), c\rho)$. The following geometric lemmas assert that if there is at least one anchor in each of B_W, B_E, B_N, B_S then $\mathcal{S}_O \subset S_{NN}$.

Denote by **B** the event that there is at least one anchor in each of B_W, B_E, B_N, B_S. The following lemma asserts that conditioned on this event, the area of intersection I_S must be contained in the disk of radius $2\sqrt{2}c\rho$ around S.

Lemma 5.1. *Conditioned on* **B**, $I_S \subset D(S, 2\sqrt{2}c\rho)$.

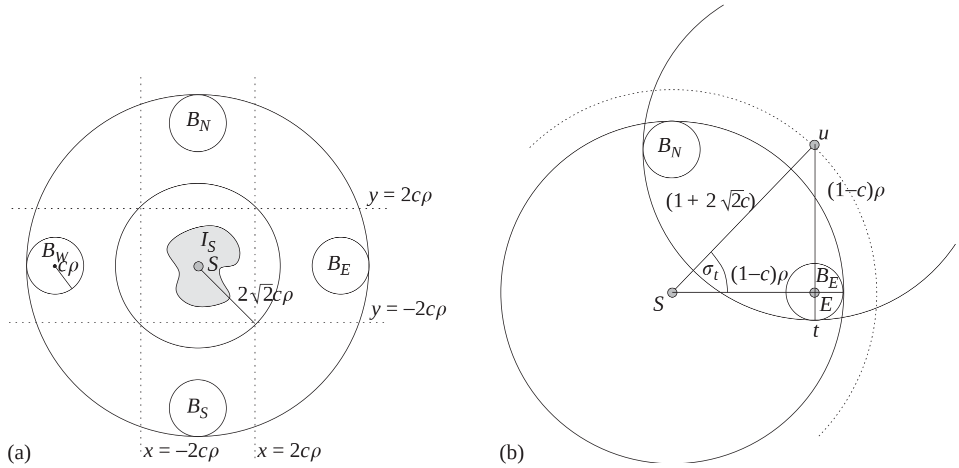

Fig. 6. If there are anchors in each of B_W, B_E, B_N, B_S then the area of intersection of the neighbors of S, I_S, is fully contained in the square centered at S and of side $4c\rho$ which is inscribed by the disk $D(S, 2\sqrt{2}c\rho)$ (a). Furthermore, the range of every anchor in $D(S, (1 + 2\sqrt{2}c)\rho)$ must fully contain at least one of the disks B_W, B_E, B_N, B_S where at the critical angles, easily shown to be $\theta_t = \frac{\pi}{4}k$, $k \in \mathbb{N}$, it must fully contain two out of them (b).

Proof. Let u be a point in I_S. By the definition of I_S there must be points $v_W \in B_W$, $v_E \in B_E$, $v_N \in B_N$ and $v_S \in B_S$ such that $\|u - v_W\|, \|u - v_E\|, \|u - v_N\|, \|u - v_S\| \leqslant \rho$. By the convexity of B_W, u must belong to the half plane $x \leqslant 2c\rho$. Likewise, by the convexity of B_E, B_N, B_S u must belong to the half planes $x \geqslant -2c\rho$, $y \geqslant -2c\rho$, $y \leqslant 2c\rho$, respectively; see Fig. 6(a). Thus, any point $u \in I_N$ must belong to the square inscribed by the boundary of $D(S, 2\sqrt{2}c\rho)$ and the lemma holds. $\square$

The following trivial corollary asserts that the anchors in $\mathcal{S}_O$ cannot be too far from S.

Corollary 5.2. *Conditioned on* **B**, $\mathcal{S}_O \subset D(S, (1 + 2\sqrt{2}c)\rho)$.

In the last geometric lemma we show that anchors in $\mathcal{S}_O$ must have at least one of B_W, B_E, B_N, B_S fully within range, so $\mathcal{S}_O \subset \mathcal{S}_{NN}$.

Lemma 5.3. *Conditioned on* **B**, $\mathcal{S}_O \subset \mathcal{S}_{NN}$.

Proof. By Corollary 5.2 every point in $\mathcal{S}_O$ must be in the annulus $D(S, (1 + 2\sqrt{2}c)\rho) \setminus D(S, \rho)$. Consider at first the points on the boundary $(\theta, (1 + 2\sqrt{2}c)\rho)$, $0 \leqslant \theta \leqslant 2\pi$, and in particular, start with $u = (0, (1 + 2\sqrt{2}c)\rho)$. It is not hard to see that in this case $B_E \subset D(u, \rho)$. Now gradually increase θ until B_E is also tangent to $\partial D(u, \rho)$ and denote this critical angle by θ_t. Mark by t the tangent point and by E the center of B_E and consider the triangle ΔSuE. By the choice of u, $\|S - u\| = (1 + 2\sqrt{2}c)\rho$. By the construction of B_E, $\|S - E\| = (1 - c)\rho$ and since t is a tangent point, $\|u - E\| = \|u - t\| - \|t - E\| = (1 - c)\rho$. Therefore, θ_t is easily derived to be $\theta_t = \pi/4$. Thus, the lemma holds all points u at the boundary with $0 \leqslant \theta \leqslant \pi/4$. The same type of argument can be used to show that the lemma holds also for $\pi/2 \geqslant \theta \geqslant \pi/4$ and so on for all θ. Notice that at the critical angles, $\theta_c = \frac{\pi}{4}k, k \in \mathbb{N}$, disk $D(u, \rho)$ fully contains 2 disks out of B_W, B_E, B_N, B_S. Thus, the lemma holds for all the boundary.

Now consider a point $u = (\theta, \rho) \in D(S, (1 + 2\sqrt{2}c)\rho)$ which is not on the boundary. Denote by u' its projection on the boundary $u' = (\theta, (1 + 2\sqrt{2}c)\rho)$, and by B' a disk out of B_W, B_E, B_S, B_W that is fully contained in $D(u', \rho)$. Since B' is also fully contained in $D(S, \rho)$ and since $D(S, (1 + 2\sqrt{2}c)\rho)$ is convex, it follows that B' must also be fully contained in $D(u, \rho)$ and the lemma holds. □

Lemma 5.4. *Conditioned on* **B**, $A_S^{\mathrm{aprx}} = A_S$.

Proof. Since S has neighbor anchors $A_S = I_S \setminus \bigcup_{s \in \mathcal{S}_O} D(s, \rho)$ and $A_S^{\mathrm{aprx}} = I_S \setminus \bigcup_{s \in \mathcal{S}_{NN}} D(s, \rho)$ (see (4), (6)). By Lemma 5.3 $\mathcal{S}_O \subset \mathcal{S}_{NN}$, and by definition of $\mathcal{S}_O$ any anchor in $\mathcal{S}_{NN}$ which affects the size of A_S is also in $\mathcal{S}_O$ so the lemma follows. □

It is not hard to show that the probability that there is at least one anchor in each of B_W, B_E, B_N, B_S is asymptotically 1. Thus, the size of the area of uncertainty computed by the approximation protocol converges in probability to that of the Arrangement protocol.

Theorem 5.5. $|A_S^{\mathrm{aprx}}| \xrightarrow{P} |A_S|$.

Proof. For any $\varepsilon > 0, \delta > 0$ define $K_0(\delta) = \rho^{-2} \ln(4/\delta)$. Given any $K \geqslant K_0(\delta)$ the probability for any specific disk out of B_W, B_E, B_S, B_N to remain empty is $(1 - \rho^2)^K$. Denoting by X the number of such empty disks, it follows that $\mathrm{E}(X) = 4(1 - \rho^2)^K$ so, using the Markov inequality,

$$\Pr[\mathbf{B}] = 1 - \Pr[X \geqslant 1] \leqslant 1 - \mathrm{E}(X) = 1 - 4(1 - \rho^2)^K \leqslant 1 - 4e^{-\rho^2 K} \leqslant 1 - \delta.$$

Thus, by Lemma 5.4, for all $K \geqslant K_0$

$$\Pr[|A_S^{\mathrm{aprx}}| - |A_S| > \varepsilon] \leqslant 1 - \Pr[\mathbf{B}] \leqslant \delta$$

and the theorem holds. □

Since $|A_S^{\mathrm{aprx}}| \leqslant |I_S|$ and $\mathrm{E}(|I_S|) < \infty$ the next corollary follows from the Lebesgue Dominated Convergence Theorem.

Corollary 5.6. $\mathrm{E}(|A_S^{\mathrm{aprx}}|) \to \mathrm{E}(|A_S|)$.

In practice $\mathrm{E}(|A_S^{\mathrm{aprx}}|)$ converges more rapidly. Fig. 7 presents simulation results for the expected uncertainty of the Approximation protocol plotted against the expectations of the Intersection and Arrangement protocols. As can been seen, the simulated expectation does converge to that of the Arrangement protocol. It should be noted that when K is small, all three expectations are dominated by the event where S has no neighbor anchors ($\mathcal{S}_N = \emptyset$). In this case the size of the area of uncertainty is the unit disk, which is typically very large compared to the other cases. As more anchors are placed, the probability of this event

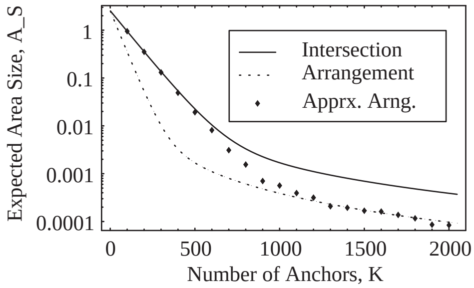

Fig. 7. Simulation results for the expected uncertainty of the Approximation algorithm are plotted together with the calculated expectations of the Intersection and Arrangement protocols ($\rho = .1$). The simulated expectation converges to that of the Arrangement protocol.

quickly diminishes to the point where it has negligible effect. This behavior, however, creates an anomaly in the shape of a phase transition in a straightforward simulation. Therefore, the data presented in Fig. 7 is the simulation results conditioned on S having neighbors analytically adjusted to take into account the case where $\mathcal{S}_N = \emptyset$.

6. Conclusions

We consider three range-free localization protocols for sensor networks and analyze their expected positioning accuracy under random sensor distribution. Assuming a small set of anchor nodes that know their position and broadcast it, we investigate at first the case where sensors position themselves at the intersection of the broadcast range of the anchors they detect (their neighbors). We then consider the positioning uncertainty when sensors avoid positioning themselves in places that are in the range of non-neighbor anchors, and an approximation to the second case where the proximity of only some local subset (the neighbor neighbors) of all the non-neighbor anchors is avoided.

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