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DISCRETE-TIME PRIORITY QUEUES WITH TWO-STATE MARKOV MODULATED ARRIVALS

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ABSTRACT

A class of discrete-time priority queueing systems with Markov modulated arrivals is considered. In these systems, N queues are served by a single server according to priorities that are preassigned to the queues. Packet arrivals are modeled as discrete-time batch processes with a distribution that depends on the state of an independent common two-state Markov chain. This allows to cover a wide range of applications in computer and communication systems when the parameters of the arrival processes are not fixed in time, but vary according to the state of the underlying Markov chain.

We derive the steady-state joint generating functions of the queue lengths distributions of this class of systems. From the latter, moments of the queue lengths as well as average time delays can be obtained.

A numerical example provides some insight into the behavior of such systems. Also, the effect of the transition rate between the states of the modulating Markov chain on the average time delays in the system is investigated for different patterns of loads on the queues of the system.

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1 INTRODUCTION

Discrete-time queueing systems have been receiving increased attention in recent years due to their usefulness in modeling and analyzing various types of communication systems. Packet switched communication networks with point-to-point links between the nodes, where data packets are of fixed length, motivated most of these models, since the fixed packet length assumption induces their discrete-time nature. These models usually assume that the input processes are independent from slot to slot and identically distributed in each slot. This assumption, though being mathematically convenient and elegant, covers only a narrow range of applications.

One modeling device that is becoming increasingly popular is *Markov modulation* where basic parameters of processes like arrival intensities or service-time distributions, are not fixed in time but may vary according to the state of an underlying discrete-time Markov chain or a continuous-time Markov process. For example, Bruneel [1] and Daigle et. al. [4] consider a single class discrete-time queueing system with Markov modulated arrivals. Burman and Smith [2] consider a single class infinite-server continuous time queueing system in which the input process is a non-homogeneous Poisson process with rate proportional to the queue length process. Other examples can be found in Lucantoni [6], and Neuts [8, 9]. In general, Markov modulation allows to generalize many standard queueing systems and also to incorporate a variety of practical models.

In this paper, we consider a class of discrete-time *priority* queueing systems with Markov modulated arrivals, where the distributions of the arrival processes to the system depend on the state of an independent common two-state Markov chain (the modulation process). This allows for generalizations that cover a wide range of applications where the arrival processes are not fixed in time but vary according to the state of the underlying Markov chain. Some examples arise in the study of packet arrivals to a local switch from N sources, where packets are transmitted ,one at a time, on a common channel according to pre-assigned priorities to the sources. The distribution of the arrival processes to the queues may change with time. Another example is, when only some sources are given permission to load their packets during some period of time and others are not given this permission. We restrict ourselves to a two-state Markov modulating chain because it greatly facilitates the analysis and the presentation. The generalization for any finite-state Markov chain is possible using the same techniques, but is very complicated. Also, the ideas and techniques used here can be extended and used for the analysis of other class of systems containing dependent discrete-time queues such as in Morrison [7] where a system with two queues in tandem has been considered and in Sidi and Segall [10] where an N-class priority queueing system has been considered.

The paper is organized as follows. In Section 2, we describe the model along with the assumptions and several definitions and notations used throughout the paper. In Section 3, we present the steady-state analysis of the class of systems under consideration. In particular, we develop a method for deriving the steady-state joint generating function of the queue lengths when the two-state modulating Markov chain is in one of it's states. We also give the ergodicity condition for the system. The corresponding moments can be derived from the generating function, and average time delays can be obtained by using Little's law. In Section 4, we give an example that provides some insight into the behavior of systems with common two-state Markov modulated arrivals.

2 MODEL DESCRIPTION

Consider a discrete-time queueing system in which the time axis is divided into intervals of equal size, referred to as slots. The slots correspond to the transmission time of a packet, and all packets are assumed to be of the same fixed size. The system consists of N queues each having an infinite buffer, and packets arrive randomly to the queues from N sources that in general may be *correlated.* Moreover, the distributions of the arrival processes depend on the state of an homogeneous discrete-time Markov chain with two states, called the *Modulating Markov Chain* (*MMC*). Transitions between the two states of the *MMC* can take place only at slot boundaries. The transition probabilities of the *MMC* are independent of the queue lengths. Let $s(t) \in \{0,1\}$, $t = 0,1,2,\ldots$, be the state of the *MMC* at time $t+$ (just after the slot boundary). The one-step transition matrix for this Markov chain is given by $\begin{bmatrix} \overline{P}_1 & P_1 \\ P_0 & \overline{P}_0 \end{bmatrix}$, where $\overline{P}_j \triangleq 1 - P_j$. We assume that $0 < P_j \leq 1,\ j = 0,1$, since if $P_j = 0$ for some j the case degenerates to non-modulated arrival processes analyzed in Sidi and Segall [10]. The limiting probabilities of the MMC are $\pi_j = P_j/(P_0 + P_1)$, $j = 0, 1$.

Let $A_i^j(t)$, $j = 0,1$, $i = 1,2,\ldots,N$, $t = 0,1,2,\ldots$, be the number of packets entering queue i from its corresponding source during the time interval $(t, t+1]$, given the MMC is in state j at time $t+$. The input process $\left\{A_i^j(t)\right\}_{i=1}^N$, $j = 0,1$, $t = 0,1,2,\ldots$, is assumed to be a sequence of independent random vectors with integer-valued elements. Let $T_j \triangleq \{t_1^j, t_2^j, \ldots\}$, $t_k^j \in I\!N$, be a subset of times in which the MMC is in state j, $j = 0,1$. Then, the input process $\left\{A_i^j(t)\right\}_{i=1}^N$, $t \in T_j$, is assumed to be a sequence of identically distributed random vectors. Let the corresponding probability distribution and generating function of the input process given the MMC is in state j, $j = 0,1$, be (we define $\underline{z} \triangleq (z_1, z_2, \ldots, z_N)$):

$$a_j(i_1, \ldots, i_N) = Pr\{A_1^j(t) = i_1, \ldots, A_N^j(t) = i_N\},\ i_k = 0,1,2,\ldots,\ 1 \le k \le N;$$

$$F_j(\underline{z}) = E\left\{\prod_{i=1}^N z_i^{A_i^j(t)}\right\}, \qquad |z_i| \le 1,\ i = 1,2,\ldots,N,$$

where $F_j(\underline{z})$, $j = 0,1$, is an analytic function in the polydisk $|z_i| < 1$, $i = 1,2,\ldots,N$. Furthermore, we assume that $F_j(\underline{z})$, $j = 0,1$, is analytic on $|z_i| = 1$, $i = 1,2,\ldots,N$. Note that the arrival processes considered are rather general, and allow correlation among the arrivals to the different queues within a slot.

Next we describe the departure processes from the queues. All queues are served by a single server and service can start only at the beginning of a slot. No more than one packet may be served in any given time slot. The queues are served according to a fixed priority. Specifically, queue i $(1 \le i \le N)$ is served only when queues $1, 2, \ldots, i-1$ are empty and the one at queue i is nonempty. The buffers at the queues are infinite. Finally, we assume that packets indeed arrive at every queue with nonzero probability.

3 STEADY-STATE ANALYSIS

3.1 System Evolution and Preliminaries

To describe the evolution of the contents of the queues, we need several definitions. Let $L_i(t)$, $1 \le i \le N$, $t = 0,1,2,\ldots$, be the number of packets at queue i at time

$t+$ and let $U(L_i(t))$ $(1 \leq i \leq N, t = 0,1,2,\ldots)$ be a binary-valued random variable that takes value 1 if $L_i(t) > 0$ and 0 otherwise. Let $A_i(t)$ $(1 \leq i \leq N, t = 0,1,2,\ldots)$ be the number of packets entering queue i from its corresponding source during the time interval $(t,\ t+1]$. Using these definitions, it is easy to see that the system under consideration (the queue lengths) evolves for $t = 0,1,2,\ldots$ as follows. For $1 \leq i \leq N$,

$$L_i(t+1) \quad = \quad L_i(t) + A_i(t) - U(L_i(t)) \prod_{m=1}^{i-1} [1 - U(L_m(t))] \qquad (1)$$

Clearly, $\{L_1(t),\ L_2(t),\ \ldots,\ L_N(t),\ s(t)\}$, $t = 0,1,2,\ldots$, is a vector Markov chain. We are primarily interested in the queue lengths distribution when the *MMC* is in some state. Let $I^j(t)$ $(j = 0,1,\ t = 0,1,2,\ldots)$, be a binary-valued random variable that takes value 1 if $s(t) = j$ and 0 otherwise. Assuming that the vector Markov chain is ergodic (we shall derive the condition for this later), let us consider the steady-state joint generating function of the queue lengths distribution when the *MMC* is in state j, $j = 0,1$, namely

$$G^j(\underline{z}) \triangleq \lim_{t\to\infty} E\left\{\prod_{i=1}^{N} z_i^{L_i(t)} I^j(t)\right\} \qquad (2)$$

For notational convenience, let us define the following operators on $G^j(\underline{z})$, $j = 0,1$:

$$\xi_i G^j(\underline{z}) \quad \to \quad G^j(\underline{z})\Big|_{z_1=z_2=\cdots=z_i=0}\ , \qquad 1 \leq i \leq N \qquad (3)$$

We shall sometimes denote the constants $\xi_N G^j(\underline{z})$ by $G^j(\underline{0})$. With the above notations we prove in Appendix A the following theorem.

Theorem 1 *The following holds:*

$$\begin{bmatrix} G^0(\underline{z}) \\ G^1(\underline{z}) \end{bmatrix} = \frac{\sum_{i=1}^{N}(z_{i+1}^{-1} - z_i^{-1}) \begin{bmatrix} B_0(\underline{z}) & P_0F_1(\underline{z}) \\ P_1F_0(\underline{z}) & B_1(\underline{z}) \end{bmatrix} \begin{bmatrix} \xi_i G^0(\underline{z}) \\ \xi_i G^1(\underline{z}) \end{bmatrix}}{1 - z_1^{-1}[\overline{P}_0F_1(\underline{z}) + \overline{P}_1F_0(\underline{z})] + z_1^{-2}(1 - P_0 - P_1)F_0(\underline{z})F_1(\underline{z})} \qquad (4)$$

where we define $z_{N+1}^{-1} \triangleq 1$ *and* $B_j(\underline{z}) = F_j(\underline{z})[\overline{P}_{1-j} - (1 - P_0 - P_1)z_1^{-1}F_{1-j}(\underline{z})]$, $j = 0,1$.

In (4), we encounter a common phenomenon in dependent queues, namely, that the generating functions $G^j(\underline{z})$, $j = 0, 1$, are expressed in terms of several boundary functions. In order to uniquely determine $G^j(\underline{z})$, $j = 0, 1$, we will have to determine the boundary functions $\xi_i G^j(\underline{z})$, $j = 0, 1$, $i = 1, 2, \ldots, N$. In what follows, we develop the method for obtaining these boundary functions. Along this process we mainly use the analytic properties of the generating functions $G^j(\underline{z})$, $j = 0, 1$, within the polydisk $|z_i| < 1$, $i = 1, 2, \ldots, N$.

Before proceeding we derive the steady-state probability that all queues are empty, i.e., $G(\underline{0}) \triangleq G^0(\underline{0}) + G^1(\underline{0})$. This probability should be positive to guarantee stability.

Theorem 2 *For* $1 \leq i \leq N$, *let*

$$r_i^j \triangleq \left.\frac{\partial F_j(\underline{z})}{\partial z_i}\right|_{z_1 = z_2 = \ldots = z_N = 1}, \quad j = 0, 1 \tag{5}$$

$$r_i \triangleq \pi_0 r_i^0 + \pi_1 r_i^1$$

Then $G(\underline{0}) = 1 - \sum_{i=1}^{N} r_i$.

Proof: For $1 \leq i \leq N - 1$ define

$$G_i(1) \triangleq \left[\xi_i G^0(\underline{z}) + \xi_i G^1(\underline{z})\right]_{z_{i+1} = z_{i+2} = \cdots = z_N = 1} \tag{6}$$

If we substitute $z_j = 1$ for $j = 1, 2, \ldots, i-1, i+1, \ldots, N$ and let $z_i \to 1$ in (4), then using the normalization condition $G^0(\underline{z})|_{z_1 = z_2 = \cdots = z_N = 1} = \pi_0$, we obtain (using L'Hôpital's theorem) the following set of equations:

$$\begin{aligned} G_1(1) &= 1 - r_1 \\ G_i(1) - G_{i-1}(1) &= -r_i \quad , \quad 2 \leq i \leq N - 1 \\ G_{N-1}(1) - G(\underline{0}) &= r_N \end{aligned} \tag{7}$$

Here we have a set of N linear equations with the N constants $G_i(1)$, $1 \leq i \leq N-1$, and $G(\underline{0})$ unknown. By subsequent substitutions we obtain:

$$G_i(1) = 1 - \sum_{n=1}^{i} r_n \quad ; \quad G(\underline{0}) = 1 - \sum_{i=1}^{N} r_i \tag{8}$$

and the theorem is proved.

Note that Theorem 2 implies that the condition for steady-state is $\sum_{i=1}^{N} r_i < 1$.

Next, we obtain the boundary functions $\xi_i G^j(\underline{z})$, $j = 0, 1$, $i = 1, 2, \ldots, N$. In order to facilitate the analysis and the presentation, we first consider the case of two queues and then proceed to the general case of N queues.

3.2 A Two-Node System

Equation (4) for $N = 2$ implies:

$$\begin{bmatrix} G^0(z_1, z_2) \\ G^1(z_1, z_2) \end{bmatrix} = \begin{bmatrix} B_0(z_1, z_2) & P_0 F_1(z_1, z_2) \\ P_1 F_0(z_1, z_2) & B_1(z_1, z_2) \end{bmatrix} \times \begin{bmatrix} G^0(0, z_2) & G^0(0,0) \\ G^1(0, z_2) & G^1(0,0) \end{bmatrix} \begin{bmatrix} z_2^{-1} - z_1^{-1} \\ 1 - z_2^{-1} \end{bmatrix} K^{-1}(z_1, z_2) \tag{9}$$

where $B_j(z_1, z_2) = F_j(z_1, z_2)[\overline{P}_{1-j} - (1 - P_0 - P_1) z_1^{-1} F_{1-j}(z_1, z_2)]$, $j = 0, 1$ and $K(z_1, z_2) \triangleq 1 - z_1^{-1}[\overline{P}_0 F_1(z_1, z_2) + \overline{P}_1 F_0(z_1, z_2)] + z_1^{-2}(1 - P_0 - P_1) F_0(z_1, z_2) F_1(z_1, z_2)$.

In order to proceed, we shall need the following lemma.

Lemma 1 *For a given $|z_2| < 1$, the following equation in z_1,*

$$z_1^2 - z_1[\overline{P}_0 F_1(z_1, z_2) + \overline{P}_1 F_0(z_1, z_2)] + (1 - P_0 - P_1) F_0(z_1, z_2) F_1(z_1, z_2) = 0 \tag{10}$$

has exactly two solutions $z_{1,k} = z_{1,k}(z_2)$, $k = 1, 2$, in the unit circle $|z_1| < 1$.

Proof: The assumption that packets do arrive at all queues implies that there exists a state $s' \in \{0, 1\}$ such that $a_{s'}(i_1, i_2) > 0$ for some i_1 and some $i_2 > 0$. Therefore, for $|z_1| = 1$ and $|z_2| < 1$

$$\begin{aligned} |F_{s'}(z_1, z_2)| &= \left| \sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} a_{s'}(i_1, i_2) z_1^{i_1} z_2^{i_2} \right| \le \sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} a_{s'}(i_1, i_2) |z_1|^{i_1} |z_2|^{i_2} \\ &< \sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} a_{s'}(i_1, i_2) = 1 \end{aligned} \tag{11}$$

In order to proceed with the proof of the lemma we shall need the following claim.

Claim 1 *For given* $|z_2| < 1$, *the following equation in* z_1

$$\left(z_1 - \overline{P}_0 F_1(z_1, z_2)\right)\left(z_1 - \overline{P}_1 F_0(z_1, z_2)\right) = 0$$

has exactly two solutions in the unit circle $|z_1| < 1$.

The proof of the claim is as follows. Let $|z_1| = 1$ and $|z_2| < 1$, then

$$|\overline{P}_0 F_1(z_1, z_2)| \leq \overline{P}_0 < 1 = |z_1| \quad ; \quad |\overline{P}_1 F_0(z_1, z_2)| \leq \overline{P}_1 < 1 = |z_1|$$

Hence, applying Rouche's theorem (see Appendix B), the following equations in z_1,

$$z_1 - \overline{P}_0 F_1(z_1, z_2) = 0 \quad ; \quad z_1 - \overline{P}_1 F_0(z_1, z_2) = 0$$

has each a unique solution in the unit circle $|z_1| < 1$. Therefore the claim is proved.

To prove the lemma, let $|z_1| = 1$ and $|z_2| < 1$. Rearranging (10), we obtain

$$\left(z_1 - \overline{P}_0 F_1(z_1, z_2)\right)\left(z_1 - \overline{P}_1 F_0(z_1, z_2)\right) - P_0 P_1 F_0(z_1, z_2)\, F_1(z_1, z_2) = 0$$

Then,

$$\begin{aligned}
&\left|\left(z_1 - \overline{P}_0 F_1(z_1, z_2)\right)\left(z_1 - \overline{P}_1 F_0(z_1, z_2)\right)\right| \\
&= \left(|z_1| - \overline{P}_0 |F_1(z_1, z_2)|\right)\left|z_1 - \overline{P}_0 F_1(z_1, z_2)\right|\left|z_1 - \overline{P}_1 F_0(z_1, z_2)\right| \\
&\geq \left(|z_1| - \overline{P}_0 |F_1(z_1, z_2)|\right)\left(|z_1| - \overline{P}_1 |F_0(z_1, z_2)|\right) \\
&> (1 - \overline{P}_0)(1 - \overline{P}_1) = P_0 P_1 > P_0 P_1 |F_0(z_1, z_2) F_1(z_1, z_2)|
\end{aligned}$$

where, in the first inequality we have used the triangle inequality and in the last two inequalities we have used (11). Hence, applying Rouche's theorem, the proof of the lemma is completed.

We are now ready to determine the boundary functions $G^j(0, z_2)$, $j = 0, 1$, and the constants $G^j(0, 0)$, $j = 0, 1$.

3.2.1 Determination of the Boundary Functions $G^j(0, z_2)$, $j = 0, 1$.

Let $z_{1,k}$, $k = 1, 2$, denote the two solutions of (10). Here we assume that $z_{1,1} \neq z_{1,2}$. We also assume that $P_0 + P_1 \neq 1$ (the case where $P_0 + P_1 = 1$ is referred later).

Since $G^0(z_1,z_2)$ is analytic within the polydisk $|z_i|<1,\ i=1,2$, then in this disk whenever the denominator of $G^0(z_1,z_2)$ vanishes, the numerator must also vanish. Since the denominator of $G^0(z_1,z_2)$ vanishes at $z_{1,k},\ k=1,2$, we have from (9) the following two equations

$$B_0(z_{1,i},z_2)\cdot\{(z_2^{-1}-z_{1,i}^{-1})G^0(0,z_2)+(1-z_2^{-1})G^0(0,0)\} \qquad (12)$$
$$+P_0F_1(z_{1,i},z_2)\{(z_2^{-1}-z_{1,i}^{-1})G^1(0,z_2)+(1-z_2^{-1})G^1(0,0)\}=0,\quad i=1,2$$

from which the boundary functions $G^j(0,z_2),\ j=0,1$, can be expressed as a function of $G^j(0,0),\ j=0,1$. (For $z_{1,1}=z_{1,2}$, $G^0(z_1,z_2)$ is analytic in the disk $|z_i|<1,\ i=1,2$, and therefore the numerator must have a zero of order 2 at $z_{1,1}$, and one obtains two equations from which $G^j(0,z_2),\ j=0,1$ are expressed in terms of $G^j(0,0),\ j=0,1$).

It is important to note that when the *numerator* of $G^0(z_1,z_2)$ vanishes at $z_{1,k}$, $k=1,2$, also the numerator of $G^1(z_1,z_2)$ vanishes at these values. Hence, the two equations obtained from $G^1(z_1,z_2)$ (in a similar way as from $G^0(z_1,z_2)$ above) are the same as (12).

For the case $P_0+P_1=1$ (the non-modulated case), we have from (9):

$$\begin{bmatrix} G^0(z_1,z_2) \\ G^1(z_1,z_2) \end{bmatrix} = \begin{bmatrix} P_0F_0(z_1,z_2) & P_0F_1(z_1,z_2) \\ P_1F_0(z_1,z_2) & P_1F_1(z_1,z_2) \end{bmatrix} \qquad (13)$$
$$\times \begin{bmatrix} G^0(0,z_2) & G^0(0,0) \\ G^1(0,z_2) & G^1(0,0) \end{bmatrix} \times \begin{bmatrix} z_2^{-1}-z_1^{-1} \\ 1-z_2^{-1} \end{bmatrix} \times K^{-1}(z_1,z_2)$$

where $K(z_1,z_2)\triangleq 1-z_1^{-1}[\overline{P}_0F_1(z_1,z_2)+\overline{P}_1F_0(z_1,z_2)]$

Note that, in this case,

$$P_0G^1(z_1,z_2) = P_1G^0(z_1,z_2) \qquad (14)$$

By (11) and Rouche's theorem we can prove that for given $|z_2|<1$, the following equation in z_1,

$$z_1 - [\overline{P}_0F_1(z_1,z_2)+\overline{P}_1F_0(z_1,z_2)] = 0$$

has a unique solution in the unit circle $|z_1|<1$. Let $z_{1,1}=z_{1,1}(z_2)$ denote this unique solution. Then, by similar arguments as before we have the following equation,

$$F_0(z_{1,1},z_2)\cdot\{(z_2^{-1}-z_{1,1}^{-1})G^0(0,z_2)+(1-z_2^{-1})G^0(0,0)\}$$
$$+\ F_1(z_{1,1},z_2)\{(z_2^{-1}-z_{1,1}^{-1})G^1(0,z_2)+(1-z_2^{-1})G^1(0,0)\}\ =\ 0 \quad (15)$$

Now, from (14) for $z_1 = 0$ and (15) the boundary functions $G^j(0, z_2)$, $j = 0, 1$, can be expressed as a function of $G^j(0,0)$, $j = 0, 1$.

3.2.2 Determination of the Constants $G^j(0,0)$, $j = 0, 1$

To determine the constants $G^j(0.0)$, $j = 0, 1$, we shall need the following lemma.

Lemma 2 *Under the assumption of steady-state ($r_1 + r_2 < 1$), the following equation in z,*

$$z^2 - z[\overline{P}_0F_1(z,z) + \overline{P}_1F_0(z,z)] + (1-P_0-P_1)F_0(z,z)F_1(z,z) = 0 \qquad (16)$$

has exactly two solutions $z_k = \sigma_k$, $k = 1,2$, within the unit circle $|z| \leq 1$.

The proof of Lemma 2 appears in Appendix B.

One of the solutions of (16) is clearly $\sigma_2 = 1$. The other solution is denoted by σ_1, where $\sigma_1 \neq 1$ and is a real number (otherwise if (16) has a zero of order 2 at $z = 1$ then, it can be shown that the steady-state condition is violated).

Theorem 3 *For $P_0 + P_1 \neq 1$, let σ_1 be the unique solution (which is $\neq 1$) of (16). Then the following holds:*

$$\begin{bmatrix} G^0(0,0) \\ G^1(0,0) \end{bmatrix} = \frac{G(0,0)}{F_0(\sigma_1,\sigma_1)[\overline{P}_1-(1-P_0-P_1)\sigma_1^{-1}F_1(\sigma_1,\sigma_1)] - P_0F_1(\sigma_1,\sigma_1)}$$
$$\times \begin{bmatrix} -P_0F_1(\sigma_1,\sigma_1) \\ F_0(\sigma_1,\sigma_1)[\overline{P}_1-(1-P_0-P_1)\sigma_1^{-1}F_1(\sigma_1,\sigma_1)] \end{bmatrix}$$

where $G(0,0) = 1 - r_1 - r_2$ (see Theorem 2 for $N = 2$).

Proof: Since $G^0(z,z)$ is analytic within the disk $|z| \leq 1$ then within this disk when ever the denominator of $G^0(z,z)$ vanishes, the numerator must also vanish. Since the denominator of $G^0(z,z)$ vanishes at σ_1 we have from (9) that

$$F_0(\sigma_1,\sigma_1)[\overline{P}_1-(1-P_0-P_1)\sigma_1^{-1}F_1(\sigma_1,\sigma_1)]\,G^0(0,0)+P_0F_1(\sigma_1,\sigma_1)\,G^1(0,0)=0 \quad (17)$$

From (17) and the fact that $G(0,0) = G^0(0,0) + G^1(0,0)$, we have two equations from which the constants $G^j(0,0)$, $j = 0,1$ are obtained. It is important to note that $F_j(\sigma_1,\sigma_1) \neq 0$ for $j = 0$ and $j = 1$, otherwise $\sigma_1 = 0$ and the system is unstable ($a_j(0,0) = 0$ for $j = 0,1$). The above two equations have only one solution, $G^j(0,0)$, $j = 0,1$, otherwise it can be shown that $G(0,0) = 0$, a contradiction with the steady-state assumption. This completes the proof of Theorem 3.

For the case $P_0 + P_1 = 1$ the relation $P_0G^1(z_1,z_2) = P_1G^0(z_1,z_2)$, for $z_1 = z_2 = 0$ is used for the calculation of the constants $G^j(0,0)$, $j = 0,1$.

After the constants $G^j(0,0)$, $j = 0,1$ have been obtained, we obtain the boundary functions $G^j(0,z_2)$, $j = 0,1$, from (12). Then the joint generating functions $G^j(z_1,z_2)$, $j = 0,1$, are uniquely determined.

Having obtained the joint generating functions $G^j(z_1,z_2)$, $j = 0,1$, we can derive, at least in principle, any moment of the queue lengths at the queues when the *MMC* is in state $j \in \{0,1\}$. Specifically, if we denote by L_i^j, $i = 1,2$, $j = 0,1$, the average queue length at queue i when the *MMC* is in state j in steady-state, then $L_i^j = \frac{\partial G^j(z_1,z_2)}{\partial z_i}\Big|_{z_1=z_2=1}$. Denote by L_i and r_i, $i = 1,2$, the average queue length and the arrival rate, respectively, at queue i in steady-state, then $L_i = L_i^0 + L_i^1$, $r_i = r_i^0 + r_i^1$ where r_i^j, $j = 0,1$, was defined in (5). Assuming that packets arrive at the queues only at the end of a slot and then using Little's law [5] we may also obtain the average time delays at queue i (the average number of slots that a packet spends in the queue from its arrival epoch until it departs the system), denoted by T_i from $T_i = L_i/r_i$. The total average time delay in the system is obtained by applying Little's law to the whole system, and is given by $T = (L_1 + L_2)/(r_1 + r_2)$. If the arrival of a packet within a slot is uniformly distributed, then one should add an extra half a slot to the delays that we are computing. The total average delay T is clearly a function of the transition probabilities P_0 and P_1 of the *MMC*. For $P_0 = P_1 = P$, $0 < P \leq 1$, the steady-state probabilities π_j, $j = 0,1$, are equal. In this case it is of interest to investigate the influence of the transition rate between the states of the *MMC* on the total average delay T. This will be demonstrated by the examples given in Section 4.

3.3 The N-Node System

In order to uniquely determine $G^j(\underline{z})$, $j = 0,1$ for the N-node system we have to determine the boundary functions $\xi_i G^j(\underline{z})$, $j = 0,1$, $i = 1,2,\ldots,N$.

By a straightforward expansion of Lemma 2 and Theorem 3 from Section 3.2 to a system with N queues, we can prove that for $P_0 + P_1 \neq 1$ the following holds:

$$\begin{bmatrix} G^0(\underline{0}) \\ G^1(\underline{0}) \end{bmatrix} = \frac{G(\underline{0})}{F_0(\underline{\sigma})[\overline{P}_1 - (1-P_0-P_1)\sigma_1^{-1}F_1(\underline{\sigma})] - P_0F_1(\underline{\sigma})} \times \begin{bmatrix} -P_0F_1(\underline{\sigma}) \\ F_0(\underline{\sigma})[\overline{P}_1 - (1-P_0-P_1)\sigma_1^{-1}F_1(\underline{\sigma})] \end{bmatrix} \tag{18}$$

where $G(\underline{0}) = 1 - \sum_{i=1}^{N} r_i$ and $\underline{\sigma}$ denotes the vector $\underline{z}$ with $z_i = \sigma_1$, $1 \leq i \leq N$, where $\sigma_1 \neq 1$ is the unique solution within the unit circle $|z_1| \leq 1$ of the equation,

$$z^2 - z[\overline{P}_0F_1(\underline{z}_N) + \overline{P}_1F_0(\underline{z}_N)] + (1-P_0-P_1)F_0(\underline{z}_N)F_1(\underline{z}_N) = 0$$

The vector $\underline{z}_N$ denotes the vector $\underline{z}$ with $z_i = z$, $1 \leq i \leq N$. Also notice that σ_1 is a real number. For the case $P_0 + P_1 = 1$ we have, as in the two queue system, that $P_0G^1(\underline{z}) = P_1G^0(\underline{z})$, from which the constants $G^j(\underline{0})$, $j = 0,1$, can be obtained.

3.3.1 Determination of the Boundary Terms $\xi_i G^j(z)$, $j = 0,1$, $1 \leq i \leq N-1$

To proceed we need the equivalent to Lemma 1 for the N-node system.

Lemma 3 *Let $\underline{z}_i$ denote the vector $\underline{z}$ with z_l, $1 \leq l \leq i$ replaced by z_1. Then for each i, $1 \leq i \leq N-1$, and for given $|z_l| < 1$, $i < l \leq N$, the following equation in z_1,*

$$z_1^2 - z_1[\overline{P}_0F_1(\underline{z}_i) + \overline{P}_1F_0(\underline{z}_i)] + (1-P_0-P_1)F_0(\underline{z}_i)F_1(\underline{z}_i) = 0 \tag{19}$$

has exactly two solutions $z_{1,k}^{(i)} = z_{1,k}^{(i)}(z_{i+1}, z_{i+2}, \ldots, z_N)$, $k = 1,2$, in the unit circle $|z_1| < 1$.

The proof of Lemma 3 is very similar to the proof of Lemma 1 in Section 3.2, and will not be given here.

Now, for $1 \leq l \leq N-1$ we have immediately the following corollary:

Corollary 1 *Let $z_{1,k}^{(l)}$, $k = 1,2$, denote the two solutions of (19) with i replaced by l. We assume that $z_{1,1}^{(l)} \neq z_{1,2}^{(l)}$ and $P_0 + P_1 \neq 1$. Then, we have for $k = 1,2$*

$$\sum_{i=l}^{N}(z_{i+1}^{-1} - z_i^{-1})[F_0(\underline{z}_l)(\overline{P}_1 - (1 - P_0 - P_1)z_1^{-1}F_1(\underline{z}_l))\xi_i G^0(\underline{z})$$

$$+ P_0F_1(\underline{z}_l)\xi_i G^1(\underline{z})]\Big|_{z_1=z_l=z_{1,k}^{(l)}} = 0 \qquad (20)$$

This is true since $G^0(\underline{z}_l)$, $1 \leq l \leq N$, is analytic in the polydisk $|z_i| < 1$, $1 \leq i \leq N$. Then in this polydisk whenever the denominator of $G^0(\underline{z}_l)$ vanishes, the numerator must also vanish. Since the denominator of $G^0(\underline{z}_l)$ vanishes at $z_{1,k}^{(l)}$, $k = 1,2$, then (20) follows from (4). Hence, in (20) we have two equations from which the boundary functions $\xi_l G^j(\underline{z})$, $j = 0,1$, can be expressed in terms of all the boundary functions $\xi_i G^j(\underline{z})$, $j = 0,1$, $l+1 \leq i \leq N$.

Since $G^j(\underline{0})$, $j = 0,1$ has already been obtained in (18), then by backward recursive substitutions of $\xi_l G^j(\underline{z})$, $j = 0,1$, $l = N-1, N-2, \ldots, 1$, in (20) we obtain $\xi_i G^j(\underline{z})$, $j = 0,1$, $i = N-1, N-2, \ldots, 1$, in this order. Thus all the required boundary terms have been obtained, and the joint generating functions $G^j(\underline{z})$, $j = 0,1$, have been uniquely determined.

For the case $P_0 + P_1 = 1$, we have from (4) that $P_0G^1(\underline{z}) = P_1G^0(\underline{z})$, from which it follows that,

$$P_0\xi_i G^1(\underline{z}) = P_1\xi_i G^0(\underline{z}), \quad 1 \leq i \leq N$$

This case and the case where $z_{1,1}^{(l)} = z_{1,2}^{(l)}$ for some $1 \leq l \leq N-1$, are similar to the equivalent cases in the two-queue system and are a straightforward extensions of them.

4 NUMERICAL RESULTS

In this section, we use a simple example that provides some insight into the behavior of a priority queueing system with Markov modulated arrival processes. The example network consists of a two queue system. The joint generating functions of the arrival processes to the queues of the system, given the modulated Markov chain is in some state j, $j = 0,1$, is given by:

$$F_j(z_1, z_2) = \alpha_{12}^j z_1^2 + \alpha_1^j z_1 + \alpha_2^j z_2 + \alpha^j z_1 z_2 + 1 - \alpha_{12}^j - \alpha_1^j - \alpha_2^j - \alpha^j$$

i.e., during each slot, given the *MMC* is in state j, two packets arrive to queue 1 with probability α_{12}^j. With probability $\alpha_i^j, i = 1, 2$, a packet arrives to queue i. With probability α^j a packet arrives to both queues 1 and 2 and, with probability $1 - \alpha_{12}^j - \alpha_1^j - \alpha_2^j - \alpha^j$ no packet arrives to the system. The steady-state condition is $\pi_0[\alpha_1^0 + \alpha_2^0 + 2(\alpha_{12}^0 + \alpha^0)] + \pi_1[\alpha_1^1 + \alpha_2^1 + 2(\alpha_{12}^1 + \alpha^1)] < 1$. Let $T_i, i = 1, 2$, denote the average time delay at queue i, and let T denote the average time delay in the system. For $\alpha_{12}^0 = \alpha_{12}^1 = \alpha_1^1 = \alpha_2^1 = 0.1, \alpha^0 = \alpha^1 = 0.05, P_0 = 0.9, P_1 = 0.15$ the quantities T_1, T_2 and T are plotted in Fig. 1 as a function of α_1^0 for $\alpha_2^0 = 0.1$, and in Fig. 2 as a function of α_2^0 for $\alpha_1^0 = 0.1$. It can be seen from Fig. 1 that the average time delays T_1, T_2 and T increases as α_1^0 increases, except that for small values of α_1^0, T_1 and T decreases as α_1^0 increases. Similar behavior is observed in Fig. 2 except that T_1 is the same for all values of α_2^0 because queue 1 isn't affected by queue 2.

The effect of the transition rate between the states of the *MMC* on the average time delay in the system is demonstrated in Fig. 3, where the average time delay T is plotted as a function of $P_0 = P_1 = P$ for $\alpha_1^0 = 0.3, \alpha_{12}^0 = 0.4, \alpha_2^0 = \alpha^0 = \alpha_1^1 = \alpha_2^1 = 0.1, \alpha_{12}^1 = \alpha^1 = 0.05$. From Fig. 3 it can be seen that the average time delay T is a monotonically decreasing function of P, and for $P = 1$ this delay is minimized. Note that, in the last example, the average arrival rate to the system, given the *MMC* is in state 0 (1.4), is significantly larger than the average arrival rate to the system, given the *MMC* is in state 1 (0.4). Hence, the effect of the transition rate on the average delay in the system is obviously seen in Fig. 3. Intuitively, by fast transitions between the two states of the *MMC*, in our example, we avoid the built of very long queues in the nodes of the system. Next, we demonstrate the effect of the transition rate between the states of the *MMC* on the average time delay in the system T for several differences of the loads in the two states of the *MMC*. In Fig. 4, the average time delay in the system, T, is plotted as a function of $P_1 = P_0 = P$ for $\alpha_1^0 = \alpha_1^1 = \alpha^0 = \alpha^1 = 0.01, \alpha_{12}^0 = 0.4, \alpha_2^0 = \alpha_2^1 = 0.1$ and for various values of α_{12}^1. It can be seen from Fig. 4 that the *decreasing rate* of T as a function of P, decreases as α_{12}^1 increases, i.e., as the difference between the arrival rates to the system in states 0 and 1 of the *MMC* decreases.

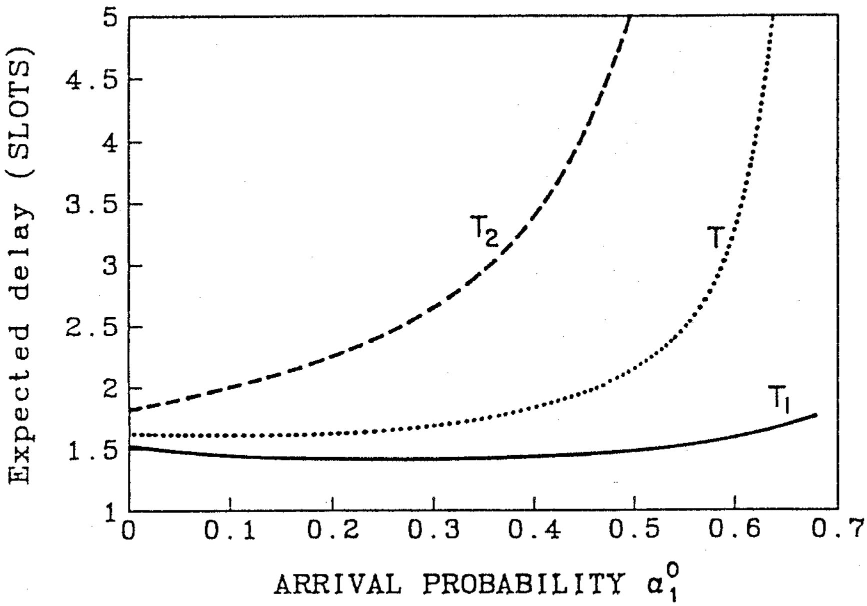

Figure 1: Average delays versus arrival probability to node 1, α_1^0.

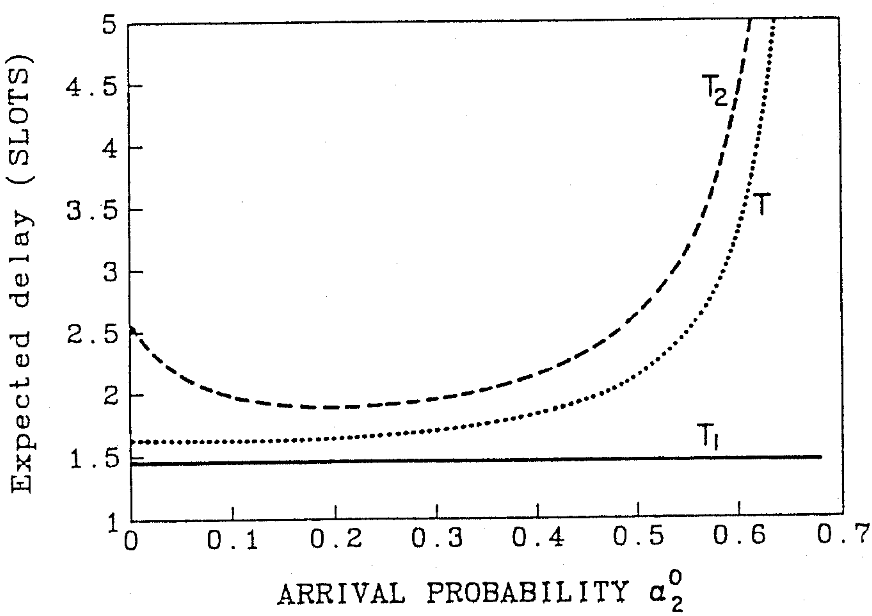

Figure 2: Average delays versus arrival probability to node 2, α_2^0.

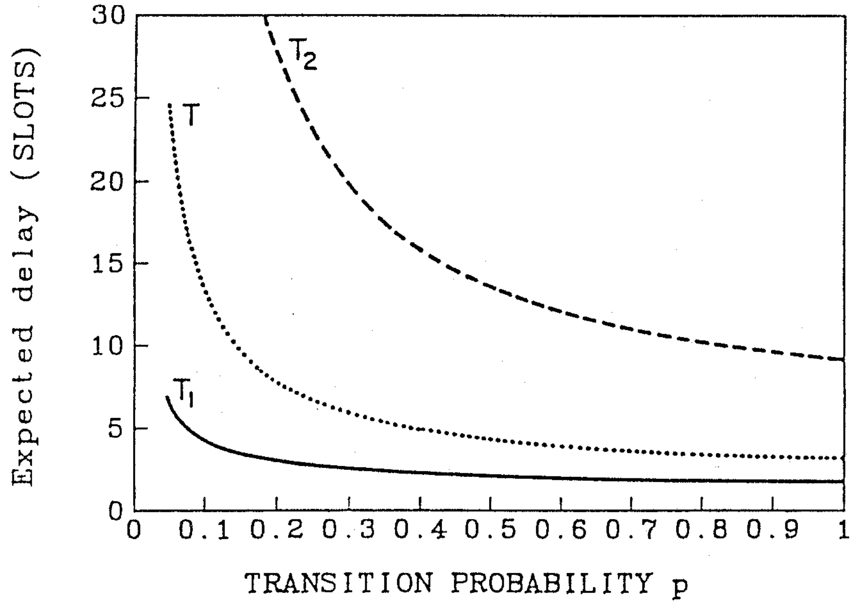

Figure 3: Average delays versus transition probability, P.

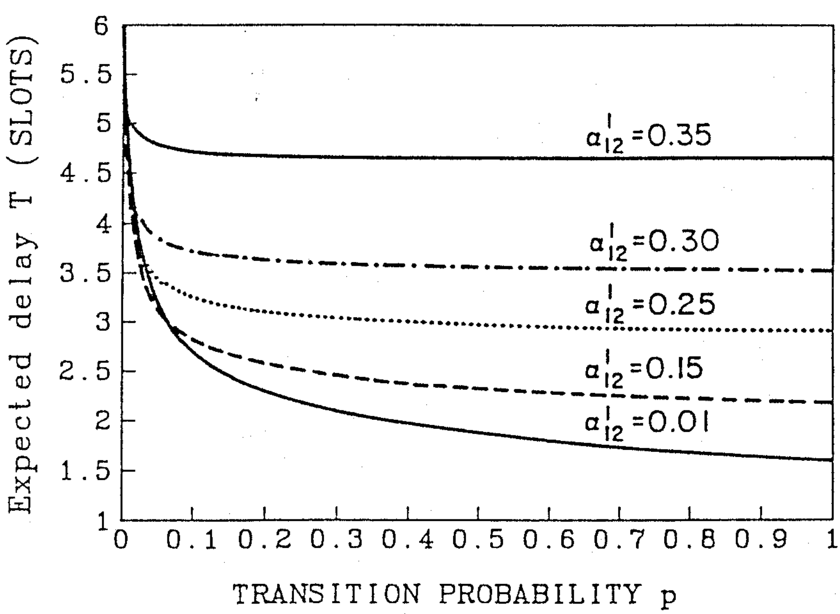

Figure 4: Average delays versus transition probability, P.

Appendix A

Proof of Theorem 1.

Consider the evolution equation (1) and let $G_t^j(\underline{z}) = E\left\{\prod_{i=1}^N z_i^{L_i(t)} \cdot I^j(t)\right\}$, $j = 0, 1$. Then,

$$
\begin{aligned}
G_{t+1}^0(\underline{z}) &= E\left\{\prod_{i=1}^N z_i^{L_i(t+1)} \cdot I^0(t+1)\right\} \qquad (21)\\
&= Pr\{s(t)=0\} \cdot E\left\{\prod_{i=1}^N z_i^{L_i(t+1)} I^0(t+1)\middle| (s(t)=0)\right\}\\
&+ Pr\{s(t)=1\} \cdot E\left\{\prod_{i=1}^N z_i^{L_i(t+1)} I^0(t+1)\middle| (s(t)=1)\right\}\\
&= \overline{P}_1 F_0(\underline{z}) Pr\{s(t)=0\} E\left\{\prod_{i=1}^N z_i^{L_i(t)-U(L_i(t))\prod_{m=1}^{i-1}[1-U(L_m(t))]}\middle| (s(t)=0)\right\}\\
&+ P_0 F_1(\underline{z}) Pr\{s(t)=1\} E\left\{\prod_{i=1}^N z_i^{L_i(t)-U(L_i(t))\prod_{m=1}^{i-1}[1-U(L_m(t))]}\middle| (s(t)=1)\right\}
\end{aligned}
$$

where in (21) we used (1) and the fact that the state of the MMC at time $t+1$ and the vector of arrival processes $\{A_i(t)\}_{i=1}^N$, given that the MMC is in state j at time t, both are independent of the queue lengths. Now, for $0 \leq k \leq N$ let the event that $L_i(t) = 0$ for $1 \leq i \leq k$ and $L_{k+1}(t) > 0$ be denoted by $\Omega_k(t)$. Then from (21) we obtain

$$
\begin{aligned}
G_{t+1}^0(\underline{z}) &= \overline{P}_1 F_0(\underline{z}) \sum_{k=0}^N Pr\{s(t)=0, \Omega_k(t)\} \qquad (22)\\
&\times \; E\left\{\prod_{i=1}^N z_i^{L_i(t)-U(L_i(t))\prod_{m=1}^{i-1}[1-U(L_m(t))]}\middle| s(t)=0, \Omega_k(t)\right\}\\
&+ \; P_0 F_1(\underline{z}) \sum_{k=0}^N Pr\{s(t)=1, \Omega_k(t)\}\\
&\times \; E\left\{\prod_{i=1}^N z_i^{L_i(t)-U(L_i(t))\prod_{m=1}^{i-1}[1-U(L_m(t))]}\middle| s(t)=1, \Omega_k(t)\right\}
\end{aligned}
$$

$$= \overline{P}_1F_0(\underline{z})\sum_{k=0}^{N} z_{k+1}^{-1}Pr\{s(t)=0,\Omega_k(t)\}E\left\{\prod_{i=k+1}^{N} z_i^{L_i(t)}\middle| s(t)=0,\Omega_k(t)\right\}$$

$$+ \; P_0F_1(\underline{z})\sum_{k=0}^{N} z_{k+1}^{-1}Pr\{s(t)=1,\Omega_k(t)\}E\left\{\prod_{i=k+1}^{N} z_i^{L_i(t)}\middle| s(t)=1,\Omega_k(t)\right\}$$

where in (22) we used the definitions of the random variables $U_i(L_i(t)), 1 \le i \le N$, and the definition $z_{N+1}^{-1} \equiv 1$. Now using the definition of $G_t^0(\underline{z})$ we have

$$\begin{aligned} G_{t+1}^0(\underline{z}) &= \overline{P}_1F_0(\underline{z})\sum_{k=0}^{N-1} z_{k+1}^{-1}\left[G_t^0(\underline{z})\mid_{z_1=z_2=\cdots=z_k=0} - G_t^0(\underline{z})\mid_{z_1=z_2=\cdots=z_{k+1}=0}\right] \\ &+ P_0F_1(\underline{z})\sum_{k=0}^{N-1} z_{k+1}^{-1}\left[G_t^1(\underline{z})\mid_{z_1=z_2=\cdots=z_k=0} - G_t^1(\underline{z})\mid_{z_1=z_2=\cdots=z_{k+1}=0}\right] \\ &+ \overline{P}_1F_0(\underline{z})G_t^0(\underline{z})\mid_{z_1=z_2=\cdots=z_N=0} + P_0F_1(\underline{z})G_t^1(\underline{z})\mid_{z_1=z_2=\cdots=z_N=0} \end{aligned} \quad (23)$$

Letting $t \to \infty$ in (23) and using the definitions in (2) and (3), we have

$$\begin{aligned} G^0(\underline{z}) &= z_1^{-1}[\overline{P}_1F_0(\underline{z})G^0(\underline{z}) + P_0F_1(\underline{z})G^1(\underline{z})] \\ &+ \sum_{k=1}^{N}(z_{k+1}^{-1} - z_k^{-1})[\overline{P}_1F_0(\underline{z})\xi_kG^0(\underline{z}) + P_0F_1(\underline{z})\xi_kG^1(\underline{z})] \end{aligned}$$

In a similar way we may obtain $G^1(\underline{z})$,

$$\begin{aligned} G^1(\underline{z}) &= z_1^{-1}[P_1F_0(\underline{z})G^0(\underline{z}) + \overline{P}_0F_1(\underline{z})G^1(\underline{z})] \\ &+ \sum_{k=1}^{N}(z_{k+1}^{-1} - z_k^{-1})[P_1F_0(\underline{z})\xi_kG^0(\underline{z}) + \overline{P}_0F_1(\underline{z})\xi_kG^1(\underline{z})] \end{aligned}$$

Therefore, rearranging the above two equations, Theorem 1 follows.

Appendix B

Proof of Lemma 2

Rearranging (16) we obtain the following equation in z:

$$(z - \overline{P}_0F_1(z,z))(z - \overline{P}_1F_0(z,z)) - P_0P_1F_0(z,z)F_1(z,z) = 0 \quad (24)$$

To prove Lemma 2 we apply Rouche's theorem [3].

Rouche's Theroem: Given two functions $f(z)$ and $g(z)$ analytic in a region R, consider a closed contour C in R; if on C we have $f(z) \neq 0$ and $|f(z)| > |g(z)|$, then $f(z)$ and $f(z) + g(z)$ have the same number of zeros within C.

To apply this theorem we identify

$$\begin{aligned} f(z) &= (z - \overline{P}_0 F_1(z,z))(z - \overline{P}_1 F_0(z,z)) \\ g(z) &= -P_0 P_1 F_0(z,z) F_1(z,z) \end{aligned}$$

The region R is the disk of radius $1+\epsilon$ (i.e. $|z| < 1+\epsilon$) for some $\epsilon > 0$. If ϵ is small enough, $f(z)$ and $g(z)$ are both analytic in R since they are analytic in $z \leq 1$ (by definition). Also, because ϵ is strictly positive we can find some δ such that $\epsilon > \delta > 0$ so that $|z| = 1+\delta$ is an appropriate contour for Rouche's theorem. If ϵ is small enough, then on this contour, $|z| = 1+\delta$, we have for $j = 0, 1$,

$$\begin{aligned} |F_j(z,z)| &= \left| \sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} a_j(i_1,i_2) z^{(i_1+i_2)} \right| \leq \sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} a_j(i_1,i_2) |z|^{(i_1+i_2)} \\ &= \sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} a_j(i_1,i_2)(1+\delta)^{(i_1+i_2)} \\ &= \sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} a_j(i_1,i_2)(1+(i_1+i_2)\delta + 0(\delta)) \\ &= 1 + (r_1^j + r_2^j)\delta + 0(\delta) \end{aligned} \tag{25}$$

where $\lim_{\delta \to 0^+} 0(\delta)/\delta = 0$. Using (25) we have

$$\begin{aligned} |f(z)| &= |z - \overline{P}_0 F_1(z,z)||z - \overline{P}_1 F_0(z,z)| \\ &\geq (|z| - \overline{P}_0 |F_1(z,z)|)(|z| - \overline{P}_1 |F_0(z,z)|) \end{aligned} \tag{26}$$

For $P_j > 0,\ j = 0,1$, we can choose $\epsilon > \delta > 0$ such that

$$P_0 > [\overline{P}_0(r_1^1 + r_2^1) - 1]\delta + 0(\delta) \qquad ; \qquad P_1 > [\overline{P}_1(r_1^0 + r_2^0) - 1]\delta + 0(\delta)$$

which implies that for $|z| = 1+\delta$,

$$\overline{P}_j |F_{1-j}(z,z)| < |z| \quad , \qquad j = 0,1\ . \tag{27}$$

Using (25)-(27) we have,

$$\begin{aligned} |f(z)| &\geq (1+\delta-\overline{P}_0(1+(r_1^1+r_2^1)\delta+0(\delta)))(1+\delta-\overline{P}_1(1+(r_1^0+r_2^0)\delta+0(\delta))) \\ &= P_0P_1+(P_0+P_1-\overline{P}_0P_1(r_1^1+r_2^1)-P_0\overline{P}_1(r_1^0+r_2^0))\delta+0(\delta)\ , \end{aligned}$$

$$|g(z)| = P_0P_1|F_0'(z,z)||F_1(z,z)| \leq P_0P_1+P_0P_1(r_1^0+r_2^0+r_1^1+r_2^1)\delta+0(\delta)\ ,$$

Using the steady-state condition $r_1+r_2<1$ (or equivalently $P_0(r_1^0+r_2^0)+P_1(r_1^1+r_2^1)<P_0+P_1$) we have from the above two equations that, $|f(z)|>|g(z)|$ implying that $f(z)$ and $f(z)+g(z)$ have the same number of roots within $|z|=1+\delta$. Now applying Rouche's theorem to $z-\overline{P}_jF_{1-j}(z,z)=0,\ j=0,1$, on the contour $|z|=1+\delta$, it follows from (27) that $f(z)$ has exactly two roots within $|z|=1+\delta$. Let $\delta\rightarrow 0^+$ then Lemma 2 is proved.

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